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不变张量技术在微分方程解的分类中的应用

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1. 闭流形上的半线性方程与不变张量技术
2. 闭 CR 流形上的（次）临界指标方程
3. Heisenberg 群上的（次）临界指标方程

1. 闭流形上的半线性方程与不变张量技术
2. 闭 CR 流形上的（次）临界指标方程
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$$\Delta u + u^{\alpha} = 0, \quad u > 0, \quad \mathbb{R}^n, \quad n \geqslant 3.$$

若 $1 < \alpha < \dfrac{n+2}{n-2}$, 无解; 若 $\alpha = \dfrac{n+2}{n-2}$, 则存在 $\lambda > 0, x_0 \in \mathbb{R}^n$, 使得

$$u(x) = \left(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}.$$

方程的由来: Yamabe 问题, Sobolev 不等式的最佳常数.

- M. Obata, 1971, J. Differential Geom.: $\alpha = \dfrac{n+2}{n-2}$ 且 $u = O(|x|^{2-n})$ $(|x| \to \infty)$
- B. Gidas, J. Spruck, 1981, Comm. Pure Appl. Math.: $1 < \alpha < \dfrac{n+2}{n-2}$
- L. Caffarelli, B. Gidas, J. Spruck, 1989, Comm. Pure Appl. Math.: $\alpha = \dfrac{n+2}{n-2}$
- 张圣容, Matthew. J. Gursky, 杨建平, 2003, New Stud. Adv. Math.: 分部积分, $\alpha = \dfrac{n+2}{n-2}$, 且 $n \geqslant 4$ 时需加体积有限条件
- 欧乾忠, arXiv:2210.05141: 分部积分, $\alpha = \dfrac{n+2}{n-2}$, $n \leqslant 5$

$$\Delta u - \lambda u + u^{\alpha} = 0,\ \ u > 0,\ \ (M^n, g)\text{是闭流形},\ \ \mathrm{Ric} \geqslant (n-1)g,\ \ n \geqslant 3.$$

当 $1 < \alpha \leqslant \dfrac{n+2}{n-2}$ 且 $0 < \lambda \leqslant \dfrac{n}{\alpha - 1}$ 时, 要么只有常数解, 要么只能在 $\alpha = \dfrac{n+2}{n-2}$, $\lambda = \dfrac{n(n-2)}{4}$, 且 (M^n, g) 等距同构与 $(\mathbb{S}^n, g_c)$ 时有非平凡解

$$u(x) = \left(\frac{\sqrt{n(n-2)}}{2\cosh t + 2(\sinh t)\langle a, x\rangle}\right)^{\frac{n-2}{2}},\ \ t \geqslant 0,\ \ a \in \mathbb{S}^n.$$

闭流形上的（次）临界指标方程

- M. Obata, 1971, J. Differential Geom.: $\alpha = \frac{n+2}{n-2}$ 且 $0 < \lambda \leqslant \frac{n(n-2)}{4}$, 还给出了 $\lambda = \frac{n(n-2)}{4}$ 时流形的刚性. 通过球极投影, 可以得到前面叙述的带有无穷远衰减的欧氏空间临界指标方程的分类结果
- M. F. Bidaut-Veron, L. Veron, 1991, Invent. Math.: $1 < \alpha < \frac{n+2}{n-2}$, 且 $0 < \lambda \leqslant \frac{n}{\alpha - 1}$
- 注: $\left.\frac{n}{\alpha-1}\right|_{\alpha=\frac{n+2}{n-2}} = \frac{n(n-2)}{4}$

旧方法：直接分部积分

在方程 $\Delta u - \lambda u + u^\alpha = 0$ 两侧同时乘以 $u^\alpha \Delta u$, 其中 α 是待定常数, 那么

$$I_1 + I_2 + I_3 = 0,$$

其中 $I_1 := \int u^\alpha (\Delta u)^2, I_2 := \int -\lambda u^{\alpha+1} \Delta u, I_3 := \int u^{\alpha+p} \Delta u$. 由散度定理,

$$I_1 = \int \left[(u^\alpha \Delta u u_j)_{,}{}^{j} - (u^\alpha \Delta u)_{,j} u^j \right] = \int \left[-\alpha u^{\alpha-1} |\nabla u|^2 \Delta u - u^\alpha u_{ij}{}^{i} u^j + u^\alpha R_{ij} u^i u^j \right]$$

$$= \int \left[u^\alpha \sum_{i,j=1}^{n} |u_{ij}|^2 + \alpha u^{\alpha-1} u_{ij} u^i u^j - \alpha u^{\alpha-1} |\nabla u|^2 \Delta u + u^\alpha R_{ij} u^i u^j \right].$$

为了配方过程计算到最佳, 引进 $E_{ij} = u_{ij} - \frac{\Delta u}{n} g_{ij}$, 则

$$\sum_{i,j=1}^{n} |E_{ij}|^2 = \sum_{i,j=1}^{n} |u_{ij}|^2 - \frac{1}{n} (\Delta u)^2.$$

将 u_{ij} 换做 E_{ij}, 继续计算 I_1:

$$I_1 = \int \left[u^\alpha \sum_{i,j=1}^{n} |E_{ij}|^2 + \alpha u^{\alpha-1} E_{ij} u^i u^j - \frac{n-1}{n} \alpha u^{\alpha-1} |\nabla u|^2 \Delta u + u^\alpha R_{ij} u^i u^j \right] + \frac{1}{n} I_1,$$

于是可以解出

$$I_1 = \frac{n}{n-1} \int \left[u^\alpha \sum_{i,j=1}^{n} |E_{ij}|^2 + \alpha u^{\alpha-1} E_{ij} u^i u^j - \frac{n-1}{n} \alpha u^{\alpha-1} |\nabla u|^2 \Delta u + u^\alpha R_{ij} u^i u^j \right].$$

类似地, 利用散度定理直接做分部积分, 并代入方程, 得

$$\begin{aligned} I_2 + I_3 =& \lambda(\alpha+1) \int u^\alpha |\nabla u|^2 - (\alpha+p) \int u^{\alpha+p-1} |\nabla u|^2 \\ =& \lambda(1-p) \int u^\alpha |\nabla u|^2 + (\alpha+p) \int u^{\alpha-1} |\nabla u|^2 \Delta u. \end{aligned}$$

旧方法: 直接分部积分

考虑 $\frac{n-1}{n}(I_1 + I_2 + I_3) = 0$, 于是

$$0 = \int \Big[u^\alpha \sum_{i,j=1}^{n} |E_{ij}|^2 + \alpha u^{\alpha-1} E_{ij} u^i u^j + \frac{n-1}{n} \lambda (1-p) u^a |\nabla u|^2 + \frac{n-1}{n} p u^{\alpha-1} |\nabla u|^2 \Delta u + u^\alpha R_{ij} u^i u^j \Big]$$

对 $\int u^{\alpha-1} |\nabla u|^2 \Delta u$ 项做分部积分:

$$\int u^{\alpha-1}|\nabla u|^2 \Delta u = -\int \left[(\alpha-1) u^{\alpha-2} |\nabla u|^4 + 2u^{\alpha-1} u_{ij} u^i u^j \right] = -\int \left[2u^{\alpha-1} E_{ij} u^i u^j + \frac{2}{n} u^{\alpha-1} |\nabla u|^2 \Delta u + (\alpha-1) u^{\alpha-2} |\nabla u|^4 \right],$$

于是可以解出

$$\int u^{\alpha-1}|\nabla u|^2\Delta u=-\frac{n}{n+2}\int\left[2u^{\alpha-1}E_{ij}u^iu^j+(\alpha-1)u^{\alpha-2}|\nabla u|^4\right].$$

代入前面的积分恒等式, 得

$$0=\int\Bigg[u^\alpha\sum_{i,j=1}^n|E_{ij}|^2+\left(\alpha-\frac{2(n-1)}{n+2}p\right)u^{\alpha-1}E_{ij}u^iu^j$$
$$-\frac{n-1}{n+2}p(\alpha-1)u^{\alpha-2}|\nabla u|^4+u^\alpha R_{ij}u^iu^j-\frac{n-1}{n}\lambda(p-1)u^\alpha|\nabla u|^2\Bigg].$$

为了配方达到最佳, 令 $L_{ij}=\frac{u_iu_j}{u}-\frac{1}{n}\frac{|\nabla u|^2}{u}g_{ij}$, 则 $\sum_{i,j=1}^n|L_{ij}|^2=\frac{n-1}{n}\frac{|\nabla u|^4}{u^2}$. 注意到 $E_i^{\ i}=0$, 因此 $u^{\alpha-1}E_{ij}u^iu^j=u^\alpha E_{ij}L^{ij}$. 将上式右端的前三项配方:

$$
\begin{aligned}
& u^\alpha \sum_{i,j=1}^n |E_{ij}|^2 + \left(\alpha - \frac{2(n-1)}{n+2}p\right) u^{\alpha-1}E_{ij}u^iu^j - \frac{n-1}{n+2}p(\alpha-1)u^{\alpha-2}|\nabla u|^4 \\
=& u^\alpha \sum_{i,j=1}^n \left| E_{ij} + \left(\frac{\alpha}{2} - \frac{n-1}{n+2}p\right) L_{ij} \right|^2 \\
& - \left[\frac{n-1}{n}\left(\frac{\alpha}{2} - \frac{n-1}{n+2}p\right)^2 + \frac{n-1}{n+2}p(\alpha-1) \right] u^{\alpha-2}|\nabla u|^4,
\end{aligned}
$$

其中 $u^{\alpha-2}|\nabla u|^4$ 项的系数为关于 α 开口向下的二次多项式:

$$
\begin{aligned}
& - \left[\frac{n-1}{n}\left(\frac{\alpha}{2} - \frac{n-1}{n+2}p\right)^2 + \frac{n-1}{n+2}p(\alpha-1) \right] \\
=& -\frac{n-1}{n}\left[\frac{\alpha^2}{4} + \frac{p\alpha}{n+2} + \frac{(n-1)^2}{(n+2)^2}p^2 - \frac{np}{n+2} \right],
\end{aligned}
$$

它在对称轴 $\alpha = -\frac{2p}{n+2}$ 处达到最大值 $\frac{n-1}{n+2}p\left(1-\frac{n-2}{n+2}p\right)$.

根据上述讨论, 取定 $\alpha = -\frac{2p}{n+2}$, 那么

$$0 = \int \Bigg[u^{-\frac{2p}{n+2}} \sum_{i,j=1}^{n} \left|E_{ij} - \frac{np}{n+2}L_{ij}\right|^2 + \frac{n-1}{n+2}p\left(1-\frac{n-2}{n+2}p\right) u^{-\frac{2p}{n+2}-2}|\nabla u|^4$$
$$+ u^{-\frac{2p}{n+2}} R_{ij}u^i u^j - \frac{n-1}{n}\lambda(p-1)u^{-\frac{2p}{n+2}}|\nabla u|^2\Bigg].$$

通过量纲分析，取 $E_{ij} = u_{ij} + c\frac{u_i u_j}{u} - \frac{1}{n}\left(\Delta u + c\frac{|\nabla u|^2}{u}\right)g_{ij}$，则

$$
\begin{aligned}
E_{ij,}{}^{i} = & \frac{n-2}{n}c\frac{E_{ij}u^i}{u} + \frac{n-1}{n}(p + \frac{n+2}{n}c)\frac{\Delta u}{u}u_j \\
& - \frac{n-1}{n}c(1 + \frac{n-2}{n}c)\frac{|\nabla u|^2}{u^2}u_j + R_{ij}u^i - \frac{n-1}{n}(p-1)\lambda u_j.
\end{aligned}
$$

根据不变张量技术，取 $c = -\frac{np}{n+2}$，有：

$$
E_{ij,}{}^{i} = -\frac{n-2}{n+2}p\frac{E_{ij}u^i}{u} + \frac{n-1}{n+2}p(1 - \frac{n-2}{n+2}p)\frac{|\nabla u|^2}{u^2}u_j.
$$

$$(E_{ij}u^j)_{,}{}^{i}=\sum_{i,j}|E_{ij}|^2+\frac{2p}{n+2}\frac{E_{ij}u^iu^j}{u}+\frac{n-1}{n+2}p(1-\frac{n-2}{n+2}p)\frac{|\nabla u|^4}{u^2}$$
$$+R_{ij}u^iu^j-\frac{n-1}{n}(p-1)\lambda|\nabla u|^2.$$

引进 u 的幂次, 将 $\dfrac{E_{ij}u^iu^j}{u}$ 消掉:

$$u^{\frac{2p}{n+2}}(u^{-\frac{2p}{n+2}}E_{ij}u^j)_{,}{}^{i}=\sum_{i,j}|E_{ij}|^2+\frac{n-1}{n+2}p(1-\frac{n-2}{n+2}p)\frac{|\nabla u|^4}{u^2}$$
$$+R_{ij}u^iu^j-\frac{n-1}{n}(p-1)\lambda|\nabla u|^2.$$

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研究 $\Delta u+f(u)=0$, 其中 $f=-f_1+f_2, f_1\in C^1(\mathbb{R}_+), f_2\in C^2(\mathbb{R}_+;\mathbb{R}_+)$.

取 $E_{ij}=u_{ij}+c\frac{f_2'(u)}{f_2(u)}u_iu_j-\frac{1}{n}\left(\Delta u+c\frac{f_2'(u)}{f_2(u)}|\nabla u|^2\right)g_{ij}$, 则

$$
\begin{aligned}
E_{ij,}{}^{i}=&\frac{n-2}{n}c\frac{f_2'(u)}{f_2(u)}E_{ij}u^i+\frac{n-1}{n}(1+\frac{n+2}{n}c)\frac{f_2'(u)}{f_2(u)}\Delta u u_j\\
&+\frac{n-1}{n}c\left[\frac{f_2''(u)}{f_2(u)}-(\frac{n-2}{n}c+1)\left(\frac{f_2'(u)}{f_2(u)}\right)^2\right]|\nabla u|^2u_j\\
&+R_{ij}u^i+\frac{n-1}{n}\left(f_1'(u)-\frac{f_2'(u)}{f_2(u)}f_1(u)\right)u_j.
\end{aligned}
$$

取 $c = -\dfrac{n}{n+2}$, 那么

$$
\begin{aligned}
& f_2(u)^{\frac{2}{n+2}} [f_2(u)^{-\frac{2}{n+2}} E_{ij} u^j]_{,}{}^{i} \\
= & \sum_{i,j=1}^{n} |E_{ij}|^2 + \frac{n-1}{n+2} \left[\frac{4}{n+2} \left(\frac{f_2'(u)}{f_2(u)} \right)^2 - \frac{f_2''(u)}{f_2(u)} \right] |\nabla u|^4 \\
& + R_{ij} u^i u^j + \frac{n-1}{n} \left(f_1'(u) - \frac{f_2'(u)}{f_2(u)} f_1(u) \right) |\nabla u|^2.
\end{aligned}
$$

Theorem (麻希南-吴天, 2024, 中国科学: 数学)

设 $f:\mathbb{R}_+\to\mathbb{R}$, 闭流形 (M^n,g) 满足 $n\geqslant 2$, $R_{ij}-(n-1)g_{ij}$ 半正定, u 是 $\Delta u+f(u)=0$ 的光滑正解. 如果 $f\geqslant 0$ 或 $f\leqslant 0$, 那么 u 为满足 $f(u)\equiv 0$ 的调和函数；如果存在 $f_1\in C^1(\mathbb{R}_+)$, $f_2\in C^2(\mathbb{R}_+;\mathbb{R}_+)$, 使得 $f=f_2-f_1$, $(\frac{f_1}{f_2})'\geqslant -\frac{n}{f_2}$, 且 $n=2$ 时, $\log f_2$ 是凹函数, $n\geqslant 3$ 时, $f_2^{\frac{n-2}{n+2}}$ 是凹函数, 那么要么 u 只能是常数, 即 f 的某个零点, 要么 $(M^n,g)=(\mathbb{S}^n,g_c)$, 且存在 $a\in\mathbb{S}^n$, $t\geqslant 0$,

Theorem (麻希南-吴天, 2024, 中国科学: 数学)

(1) 当 $n=2$ 时, 存在 $k_1>0$, $k_2\in\mathbb{R}$, 使得 $f(s)=-\frac{2}{k_1}+e^{k_1(s+k_2)}$,

$$u(x)=\frac{2}{k_1}\log\frac{2k_1^{-\frac{1}{2}}}{\cosh t+\sinh t\cdot\langle a,x\rangle_{\mathbb{R}^{n+1}}}-k_2.$$

(2) 当 $n\geqslant 3$ 时, 存在 $k_1>0$, $k_2\geqslant 0$, 使得

$$f(s)=-\frac{n(n-2)}{4}(s+k_2)+[k_1(s+k_2)]^{\frac{n+2}{n-2}},$$

$$u(x)=k_1^{-\frac{n+2}{4}}\left(\frac{\sqrt{n(n-2)}}{2\cosh t+2\sinh t\cdot\langle a,x\rangle_{\mathbb{R}^{n+1}}}\right)^{\frac{n-2}{2}}-k_2.$$

设 (M^{2n+1}, θ) 是无挠, Ricci 曲率具有正下界, 严格拟凸, 可定向的超曲面型闭 CR 流形:

- CR 结构: $\mathbb{C}TM$ 的一个复子丛 $T^{(1,0)}M$ 满足 $T^{(1,0)} \cap T^{(0,1)} = 0$, 其中 $T^{(0,1)} := \overline{T^{(1,0)}}$
- 超曲面型: $\dim_{\mathbb{R}} M = 2n+1$ 且 $\dim_{\mathbb{C}} T^{(1,0)}M = n$
- 严格拟凸: 存在一个切触形式 θ, 使得其 Levi 形式 $\langle \cdot, \cdot \rangle_\theta$ 是正定的. 切触形式是零化 $T^{(1,0)}M \oplus T^{(0,1)}M$ 的实的 1-形式, $\langle V, W\rangle_\theta := -2\sqrt{-1}\mathrm{d}\theta(V \wedge \overline{W})$, 记 $h_{i\bar{j}} = \langle Z_i, Z_j \rangle_\theta$
- 可定向、闭的概念同实流形相同
- (Webster) 挠率张量: A_{ij}, 曲率张量: $R_{i\bar{j}k\bar{l}}$, Ricci 曲率张量: $R_{i\bar{j}} := R_{i\ k\bar{j}}^{\ k}$, 不妨 $R_{i\bar{j}} \geqslant (n+1)h_{i\bar{j}}$

- 记 $T^{(1,0)}M = \mathrm{span}\{Z_i\}_{i=1}^n$, T 是 Reeb 向量场, 即 $\theta(T) = 1$, $\mathrm{d}\theta(T, X) = 0$, $\forall X \in TM$. 那么

$$TM = T^{(1,0)}M \oplus T^{(0,1)}M \oplus \mathrm{span}\{T\}.$$

事实上, M 是二步幂零群, $\{Z_i\}_{i=1}^n$, $\{Z_{\bar{i}}\}_{i=1}^n$, T 是左不变向量场

- 记 $f_{,i} = Z_i f$, $f_{,\bar{i}} = Z_{\bar{i}} f$, $f_{,0} = Tf$. 协变导数交换公式:

$$f_{,ij} = f_{,ji},\ f_{,i\bar{j}} - f_{,\bar{j}i} = 2\sqrt{-1}h_{i\bar{j}}f_{,0},\ f_{,0i} - f_{,i0} = A_{ij}f_{,}^{\ j},\ f_{,ij\bar{k}} - f_{,i\bar{k}j} = 2\sqrt{-1}h_{j\bar{k}}f_{,i0} + R_{i\ j\bar{k}}^{\ l}f_{,l}$$

- 记 $|\nabla f|^2 = f_{,i}f_{,}^{\ i}$, $\Delta f = \frac{1}{2}(f_{,i}^{\ \ i} + f_{,\ i}^{\ i})$, 那么 $f_{,i}^{\ \ i} = \Delta f + n\sqrt{-1}f_{,0}$

Heisenberg 群 $\mathbb{H}^n$ 相当于在 $\mathbb{C}^n \times \mathbb{R}$ 上配备了一个群乘法运算 $\circ$:

$$(z,t)\circ(z',t') = (z+z', t+t'+2\,\mathrm{Im}\, z\cdot\bar{z}'),\ \forall (z,t),(z',t')\in\mathbb{C}^n\times\mathbb{R}.$$

- $Z_i := \dfrac{\partial}{\partial z_i} + \sqrt{-1}\bar{z}_i\dfrac{\partial}{\partial t}$, $Z_{\bar{i}} := \dfrac{\partial}{\partial \bar{z}_i} - \sqrt{-1}z_i\dfrac{\partial}{\partial t}$, $T := \dfrac{\partial}{\partial t}$
- 是一个平坦 ($R_{\bar{i}jk\bar{l}} = 0$), 无挠, 严格拟凸, 可定向的超曲面型非紧 CR 流形
- Cauchy-Riemann 流形的齐性维数 $Q = 2n+2$

设 (M^{2n+1},θ) 是无挠, 严格拟凸, 可定向的超曲面型闭 CR 流形, $Ric_{i\bar{j}} \geqslant (n+1)h_{i\bar{j}}$,

$$\Delta_b u - \lambda u + u^{\alpha} = 0, \quad u > 0.$$

其中 $\alpha = \dfrac{n+2}{n} = \dfrac{Q+2}{Q-2}$ 为临界指标, $Q = 2n+2$ 为齐性维数.

- 王晓东, 2015, Math. Res. Lett.: $\alpha = \dfrac{n+2}{n}$ 且 $0 < \lambda \leqslant \dfrac{n^2}{4}$ 时, 要么只有常数解, 要么只能在 $\lambda = \dfrac{n^2}{4}$, 且 (M^{2n+1},θ) 等距同构与 $(\mathbb{S}^{2n+1},\theta_c)$ 时有非平凡解

$$u(z) = c_{n,s}\left|\cosh s + (\sinh s)\langle z,\xi\rangle\right|^{-n}, \quad s \geqslant 0, \quad \xi \in \mathbb{S}^{2n+1}$$

- 王晓东, 2022, Math. Z., 给出新证明

Theorem (麻希南-欧乾忠-吴天, 2023, arXiv: 2311.16428v2)

设 (M^{2n+1}, θ) 是无挠, 严格拟凸, 可定向的超曲面型闭 CR 流形, $Ric_{i\bar{j}} \geqslant (n+1)h_{i\bar{j}}$,

$$\Delta u - \lambda u + u^{\alpha} = 0, \quad u > 0,$$

那么当 $1 < \alpha < \dfrac{n+2}{n}$ 且 $0 < \lambda \leqslant \dfrac{n}{2(\alpha-1)}$ 时, $u \equiv \lambda^{\frac{1}{\alpha-1}}$.

- 这是王晓东于 2022 年提出的猜想, 我们的定理完整地解决了它. 关于猜想的提出, 具体请参考以下论文的 Conjecture 1:
  Uniqueness results on a geometric PDE in Riemannian and CR geometry revisited. Math. Z., 301(2):1299 1314, 2022.
- 注: $\dfrac{n}{2(\alpha-1)}\Big|_{\alpha=\frac{n+2}{n}} = \dfrac{n^2}{4}$.

Corollary (麻希南-欧乾忠-吴天, 2023, arXiv: 2311.16428v2)

设 (M^{2n+1}, θ) 是无挠, 严格拟凸, 可定向的超曲面型闭 CR 流形, $Ric_{i\bar{j}} \geqslant (n+1)h_{i\bar{j}}$, 那么当 $2 < q < \dfrac{2Q}{Q-2} = \dfrac{2n+2}{n}$ 时,

$$\frac{4(q-2)}{Q-2}\int_M |\nabla_b u|^2 + \int_M |u|^2 \geqslant \mathrm{vol}(M)^{1-\frac{2}{q}}\left(\int_M |u|^q\right)^{\frac{2}{q}}.$$

不等式取等当且仅当 u 为常数.

- CR 球面情形是 Rupert L. Frank, Elliott H. Lieb, 2012, Ann. Math. 的主要结论
- 注: q 相当于 $\alpha+1$, 齐性维数 $Q = 2n+2$.

称张量 $S(u)$ 是 $\{(r,s),x,y,+/-\}$ 型的, 如果它是由有限个 (r,s) 阶张量线性组合而成, 且每个张量具有 x 次幂的 u、y 阶导数, 以及每个张量的 $\sqrt{-1}$ 个数加上向量场 T 的个数是偶数/奇数, 而 $\{(r,s),x,y,+/-\}$ 叫做张量 $S(u)$ 的量纲.

- (r,s) 阶: $u_{\alpha_1\alpha_2\cdots\alpha_{r+s}}$, 其中 $\{\alpha_1,\cdots,\alpha_{r+s}\}=\{i_1,\cdots,i_r,\overline{j_1},\cdots,\overline{j_s}\}$
- x 次幂: 忽略掉 u 的导数后, 作为 u 的单项式的幂次
- y 阶导数: 具有导数的阶数, Ric 曲率视作具有二阶导数
- 值得注意的是, 向量场 T 在齐性意义下是二阶算子, 因此 u_0 视作具有二阶导数, 即 u_0 是 $\{(0,0),1,2,-\}$ 型的
- λ 在 Cauchy-Riemann 几何上（次）临界指标方程中的地位与二阶导数等同, 因此视作 $\{(0,0),0,2,+\}$ 型的

$$\{(2,0),1,2,+\}:\ D_{ij}=u_{ij}+c_1\frac{u_iu_j}{u},$$

$$\{(1,1),1,2,+\}:\ E_{i\bar{j}}=u_{i\bar{j}}+c_2\frac{u_iu_{\bar{j}}}{u}+c_3\Delta_b u h_{i\bar{j}}$$
$$+c_4n\sqrt{-1}u_0h_{i\bar{j}}+c_5\frac{|\nabla_b u|^2}{u}h_{i\bar{j}}+c_6\lambda u h_{i\bar{j}},$$

$$\{(1,0),1,3,+\}:\ G_i=n\sqrt{-1}u_{0i}-\frac{n(n+1)}{n+2}\alpha\frac{\sqrt{-1}u_0u_i}{u}-\frac{\alpha}{n+2}\frac{\Delta_b u}{u}u_i$$
$$+\frac{n\alpha}{n+2}\left(\frac{n+1}{n+2}\alpha-1\right)\frac{|\nabla_b u|^2}{u^2}u_i+(\alpha-1)\lambda u_i,$$

其中 $\{c_l\}_{l=1}^6$ 是常数.

Theorem (麻希南-欧乾忠-吴天, 2023, arXiv: 2311.16428v2)

设 (M^{2n+1},θ) 是无挠, 严格拟凸, 可定向的超曲面型闭 CR 流形, $Ric_{i\bar{j}} \geqslant (n+1)h_{i\bar{j}}$,

$$\Delta_b u - \lambda u + u^{\alpha} = 0, \quad u > 0,$$

那么当 $1 < \alpha < \dfrac{n+2}{n}$ 且 $0 < \lambda \leqslant \dfrac{n}{2(\alpha-1)}$ 时, $u \equiv \lambda^{\frac{1}{\alpha-1}}$.

分为四种情况证明:

情形 1: $1 < \alpha \leqslant \dfrac{n+2}{n+\frac{1}{2n}}, \forall n \geqslant 2$;　情形 2: $\dfrac{n+2}{n+\frac{1}{2n}} \leqslant \alpha < \dfrac{n+2}{n}, n \in \mathbb{N}^*$;

情形 3: $1.06 \leqslant \alpha < 3, n = 1$;　情形 4: $1 < \alpha \leqslant 1.06, n = 1$.

$$L_{i\bar{j}}=\frac{u_iu_{\bar{j}}}{u}-\frac{|\nabla_b u|^2}{u}h_{i\bar{j}},\quad \mathcal{R}=R_{i\bar{j}}u^iu^{\bar{j}}-\frac{2(n+1)}{n}(\alpha-1)\lambda|\nabla_b u|^2\geqslant 0.$$

不变张量:

$$D_{ij}=u_{ij}-\alpha\frac{u_iu_j}{u},$$

$$E_{i\bar{j}}=u_{i\bar{j}}-\frac{n\alpha}{n+2}\frac{u_iu_{\bar{j}}}{u}-\frac{1}{n}\left(\Delta_b u+n\sqrt{-1}u_0-\frac{n\alpha}{n+2}\frac{|\nabla_b u|^2}{u}\right)h_{i\bar{j}},$$

$$\begin{aligned}G_i=&n\sqrt{-1}u_{0i}-\frac{n(n+1)}{n+2}\alpha\frac{\sqrt{-1}u_0u_i}{u}-\frac{\alpha}{n+2}\frac{\Delta_b u}{u}u_i\\&+\frac{n\alpha}{n+2}\left(\frac{n+1}{n+2}\alpha-1\right)\frac{|\nabla_b u|^2}{u^2}u_i+(\alpha-1)\lambda u_i.\end{aligned}$$

取 $D_{ij} = u_{ij} + c_1 \frac{u_i u_j}{u}, E_{i\bar{j}} = u_{i\bar{j}} + c_2 \frac{u_i u_{\bar{j}}}{u} - \frac{1}{n}\left(\Delta_b u + n\sqrt{-1}u_0 + c_2 \frac{|\nabla_b u|^2}{u}\right) h_{i\bar{j}},$

$$
\begin{aligned}
D_{ij,}{}^{i} =& c_1 E_j + (n+2)\sqrt{-1}u_{0j} + (n+1)c_1 \frac{\sqrt{-1}u_0 u_j}{u} + (\frac{n+1}{n}c_1 + \alpha)\frac{\Delta_b u}{u}u_j \\
& - (\frac{n-1}{n}c_2 + 1)c_1 \frac{|\nabla_b u|^2}{u^2}u_j + R_{j\bar{i}}u^{\bar{i}} + (1-\alpha)\lambda u_j.
\end{aligned}
$$

$$
\begin{aligned}
E_{i\bar{j},}{}^{i} =& \frac{n-1}{n}c_2 D_{\bar{j}} - \frac{c_2}{n}E_{\bar{j}} + (n-1)\sqrt{-1}u_{0\bar{j}} + \frac{n^2-1}{n}c_2 \frac{\sqrt{-1}u_0 u_{\bar{j}}}{u} + \frac{n-1}{n}\times \\
& (\frac{n+1}{n}c_2 + \alpha)\frac{\Delta_b u}{u}u_{\bar{j}} - \frac{n-1}{n}(c_1 - \frac{c_2}{n} + 1)c_2 \frac{|\nabla_b u|^2}{u^2}u_{\bar{j}} + \frac{n-1}{n}(1-\alpha)\lambda u_{\bar{j}}.
\end{aligned}
$$

$$\frac{n+2}{-(n-1)}=\frac{(n+1)c_1}{-\frac{n^2-1}{n}c_2}=\frac{\frac{n+1}{n}c_1+\alpha}{\frac{n-1}{n}(\frac{n+1}{n}c_2+\alpha)},$$

进而 $c_1=-\alpha, c_2=-\dfrac{n\alpha}{n+2}$. 于是

$$D_{ij,}{}^{i}=-\alpha E_j+\frac{n+2}{n}G_j+2\alpha(1-\frac{n\alpha}{n+2})\frac{|\nabla_b u|^2}{u^2}u_j+R_{j\bar{i}}u^{\bar{i}}-\frac{2(n+1)}{n}(\alpha-1)\lambda u_j,$$

$$E_{i\bar{j},}{}^{i}=-\frac{n-1}{n+2}\alpha D_{\bar{j}}+\frac{\alpha}{n+2}E_{\bar{j}}-\frac{n-1}{n}G_{\bar{j}}.$$

$$\begin{aligned}G_i=&n\sqrt{-1}u_{0i}-\frac{n(n+1)}{n+2}\alpha\frac{\sqrt{-1}u_0u_i}{u}-\frac{\alpha}{n+2}\frac{\Delta_b u}{u}u_i\\&+\frac{n\alpha}{n+2}\left(\frac{n+1}{n+2}\alpha-1\right)\frac{|\nabla_b u|^2}{u^2}u_i+(\alpha-1)\lambda u_i,\end{aligned}$$

通过直接消去 G_i 项, 可以得到下列 $\{(0,0),2,4,+\}$ 型恒等式:

$$\begin{aligned}
&\mathrm{Re}[(n-1)D_{ij}u^{j}+(n+2)E_{i\bar{j}}u^{\bar{j}}]_{,}{}^{i}\\
=&(n+2)\sum_{i,j}|E_{i\bar{j}}|^2+(n-1)\sum_{i,j}|D_{ij}|^2+2\alpha E_iu^i+2(n-1)\alpha\Big(1-\frac{n\alpha}{n+2}\Big)\frac{|\nabla_b u|^4}{u^2}+(n-1)\mathcal{R}\\
=&(n+2)\sum_{i,j}\left|E_{i\bar{j}}+\frac{\alpha}{n+2}L_{i\bar{j}}\right|^2+(n-1)\left[\sum_{i,j}|D_{ij}|^2+\alpha\left(2-\frac{2n^2+1}{n(n+2)}\alpha\right)\frac{|\nabla_b u|^4}{u^2}+\mathcal{R}\right].
\end{aligned}$$

- 注: $n=1$ 时, 此恒等式退化, 失去作用.

考虑 $\{(0,0),2,6,+\}$ 型恒等式:

$$
\begin{aligned}
& u^{-\beta}\,\mathrm{Re}\Big\{u^{\beta}\Big[\Big(d_1\frac{|\nabla_b u|^2}{u}+d_2u^{\alpha}+d_3\lambda u+d_4 n\sqrt{-1}u_0\Big)D_i \\
& +\Big(e_1\frac{|\nabla_b u|^2}{u}+e_2u^{\alpha}+e_3\lambda u+e_4 n\sqrt{-1}u_0\Big)E_i-\mu n\sqrt{-1}u_0G_i\Big]\Big\}_{,}^{\;i} \\
= & \Big[d_1\frac{|\nabla_b u|^2}{u^2}+d_2u^{\alpha-1}+d_3\lambda\Big]\Big[\sum_{i,j=1}^{n}|D_{ij}|^2+2\alpha(1-\frac{n\alpha}{n+2})\frac{|\nabla_b u|^4}{u^2}+\mathcal{R}\Big] \\
& +\Big[e_1\frac{|\nabla_b u|^2}{u^2}+e_2u^{\alpha-1}+e_3\lambda\Big]\sum_{i,j=1}^{n}|E_{i\bar{j}}|^2+d_1\sum_{i=1}^{n}|D_i|^2+e_1\sum_{i=1}^{n}|E_i|^2 \\
& +\mu\sum_{i=1}^{n}|G_i|^2+(d_1+e_1)\,\mathrm{Re}\,D_iE^i-d_4\,\mathrm{Re}\,D_iG^i-e_4\,\mathrm{Re}\,E_iG^i
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathrm{Re}\left[\Delta_1 \frac{|\nabla_b u|^2}{u^2} + \Delta_2 u^{\alpha-1} + \Delta_3 \lambda + \Delta_4 \frac{n\sqrt{-1}u_0}{u}\right] D_i u^i \\
&+ \left[\Theta_1 \frac{|\nabla_b u|^2}{u^2} + \Theta_2 u^{\alpha-1} + \Theta_3 \lambda\right] E_i u^i \\
&+ \mathrm{Re}\left[\Xi_1 \frac{|\nabla_b u|^2}{u^2} + \Xi_2 u^{\alpha-1} + \Xi_3 \lambda + \Xi_4 \frac{n\sqrt{-1}u_0}{u}\right] G_i u^i .
\end{aligned}
$$

其中的系数如下:

$$
\Delta_1 = \left(\beta + \frac{3(n+1)}{n+2}\alpha - 2\right) d_1 - \frac{n-1}{n+2}\alpha e_1 + \frac{n\alpha}{n+2}\left(\frac{n+1}{n+2}\alpha - 1\right) d_4,
$$

$$
\Delta_2 = -\frac{1}{n} d_1 + (\beta + 2\alpha - 1) d_2 - \frac{n-1}{n+2}\alpha e_2 + \frac{\alpha}{n+2} d_4,
$$

$$\Delta_3 = \frac{1}{n}d_1 + (\beta+\alpha)d_3 - \frac{n-1}{n+2}\alpha e_3 + (\frac{n+1}{n+2}\alpha - 1)d_4,$$

$$\Delta_4 = \frac{1}{n}d_1 + \left(\beta + \frac{2n+3}{n+2}\alpha - 1\right)d_4 + \frac{n-1}{n+2}\alpha e_4 + \frac{n\alpha}{n+2}(\frac{n+1}{n+2}\alpha - 1)\mu,$$

$$\Theta_1 = -\alpha d_1 + \left(\beta + \frac{3n+2}{n+2}\alpha - 2\right)e_1 + \frac{n\alpha}{n+2}(\frac{n+1}{n+2}\alpha - 1)e_4,$$

$$\Theta_2 = -\frac{1}{n}e_1 - \alpha d_2 + \left(\beta + \frac{2n+3}{n+2}\alpha - 1\right)e_2 + \frac{\alpha}{n+2}e_4,$$

$$\Theta_3 = \frac{1}{n}e_1 - \alpha d_3 + \left(\beta + \frac{n+1}{n+2}\alpha\right)e_3 + (\frac{n+1}{n+2}\alpha - 1)e_4,$$

$$\Xi_1 = \frac{n+2}{n}d_1 - \frac{n-1}{n}e_1 - \frac{n\alpha}{n+2}(\frac{n+1}{n+2}\alpha - 1)\mu,$$

$$\Xi_2 = \frac{n+2}{n}d_2 - \frac{n-1}{n}e_2 - \frac{\alpha}{n+2}\mu,$$

$$\Xi_3 = \frac{n+2}{n}d_3 - \frac{n-1}{n}e_3 - (\frac{n+1}{n+2}\alpha - 1)\mu,$$

$$\Xi_4 = \frac{n+2}{n}d_4 + \frac{n-1}{n}e_4 - \beta\mu.$$

取 $d_1 = e_1 = \frac{n^2\alpha[3n+6-(n-1)\alpha]}{(2n+1)(n+2)^2}, d_2 = e_2 = \frac{n\alpha}{n+2},$

$$d_3 = e_3 = n(\frac{n+1}{n+2}\alpha - 1),\ d_4 = \frac{n}{2n+1}(3 - \frac{7n+2}{n+2}\alpha),$$

$$e_4 = \frac{n(3+\alpha)}{2n+1},\ \mu = 3,\ \beta = 1 - \alpha.$$

$$
\begin{aligned}
& u^{-\beta}\,\mathrm{Re}\left\{u^{\beta}\left[\left(d_1\frac{|\nabla_b u|^2}{u}+d_2u^{\alpha}+d_3\lambda u\right)(D_i+E_i)+n\sqrt{-1}u_0(d_4D_i+e_4E_i-3G_i)\right]\right\}_{,}^{\ i} \\
=& d_1u^{-2}\sum_{i,j,k}|D_{ij}u_{\bar{k}}+E_{i\bar{k}}u_j|^2+d_2u^{\alpha-1}\sum_{i,j}\left[|D_{ij}|^2+|E_{i\bar{j}}|^2+2\alpha(1-\frac{n\alpha}{n+2})\frac{|\nabla_b u|^4}{u^2}\right] \\
& +d_3\lambda\sum_{i,j}\left[\left|D_{ij}+\frac{\Delta_3}{2d_3}\frac{u_iu_j}{u}\right|^2+\left|E_{i\bar{j}}+\frac{\Delta_3}{2d_3}L_{i\bar{j}}\right|^2+\left(2\alpha(1-\frac{n\alpha}{n+2})-\frac{2n-1}{n}\frac{\Delta_3^2}{4d_3^2}\right)\frac{|\nabla_b u|^4}{u^2}\right] \\
& +\left[d_1\frac{|\nabla_b u|^2}{u^2}+d_2u^{\alpha-1}+d_3\lambda\right]\mathcal{R}+\mathbf{Q}_1,
\end{aligned}
$$

系数如下:

$$\Delta_1 = \frac{2n^2\alpha[(4n+5)\alpha - 3n - 6]}{(2n+1)(n+2)^2}\left(1 - \frac{n\alpha}{n+2}\right),$$

$$\Theta_1 = -\frac{6n^2\alpha(\alpha+n+2)}{(2n+1)(n+2)^2}\left(1 - \frac{n\alpha}{n+2}\right),$$

$$\Xi_1 = \frac{6n\alpha}{2n+1}\left(1 - \frac{n\alpha}{n+2}\right), \quad \Delta_3 = \Theta_3 = \frac{2n(\alpha-1)(2+n-n\alpha)}{2n+1},$$

$$\Delta_2 = \Theta_2 = \Xi_2 = \Xi_3 = \Delta_4 = \Xi_4 = 0.$$

$$\begin{aligned}\mathbf{Q}_1 =& d_1 \sum_{i=1}^{n} |D_i|^2 + d_1 \sum_{i=1}^{n} |E_i|^2 + 3 \sum_{i=1}^{n} |G_i|^2 - d_4 \operatorname{Re} D_i G^i - e_4 \operatorname{Re} E_i G^i \\ &+ \Delta_1 \frac{|\nabla_b u|^2}{u^2} \operatorname{Re} D_i u^i + \Theta_1 \frac{|\nabla_b u|^2}{u^2} E_i u^i + \Xi_1 \frac{|\nabla_b u|^2}{u^2} \operatorname{Re} G_i u^i + 2d_1\alpha(1 - \frac{n\alpha}{n+2}) \frac{|\nabla_b u|^6}{u^4},\end{aligned}$$

$$Q_1 = \begin{pmatrix} d_1 & 0 & -\frac{d_4}{2} & \frac{\Delta_1}{2} \\ 0 & d_1 & -\frac{e_4}{2} & \frac{\Theta_1}{2} \\ -\frac{d_4}{2} & -\frac{e_4}{2} & 3 & \frac{\Xi_1}{2} \\ \frac{\Delta_1}{2} & \frac{\Theta_1}{2} & \frac{\Xi_1}{2} & 2d_1\alpha(1 - \frac{n\alpha}{n+2}) \end{pmatrix}.$$

先检查 λ 项的正性. 当 $\alpha \in (\frac{n+2}{n+1}, \frac{n+2}{n})$ 时, $d_3 = n(\frac{n+1}{n+2}\alpha - 1) > 0$,

$$d_3\left(2\alpha(1-\frac{n\alpha}{n+2}) - \frac{2n-1}{n}\frac{\Delta_3^2}{4d_3^2}\right) = \frac{n}{(2n+1)^2 d_3}(1-\frac{n\alpha}{n+2})f_1(\alpha),$$

其中 $f_1(\alpha)$ 是关于 α 的多项式:

$$f_1(\alpha) = \frac{n(10n^4+35n^3+44n^2+16n-6)}{(n+2)^2}\alpha^3 - \frac{22n^4+57n^3+46n^2-8}{n+2}\alpha^2 + (14n^3+25n^2+8n-8)\alpha - (n+2)^2(2n-1),$$

那么 f_1' 是一个二次函数:

$$f_1'(\alpha) = 3n(10n^4+35n^3+44n^2+16n-6)\left(\frac{\alpha}{n+2}\right)^2 - 2(22n^4+57n^3+46n^2-8)\left(\frac{\alpha}{n+2}\right) + (14n^3+25n^2+8n-8).$$

比较二次函数 $f_1((n+2)x)$ 的对称轴与 $\frac{\alpha}{n+2}$ 的最小值 $\left(n+\frac{1}{2n}\right)^{-1}$ 的大小:

$$\frac{22n^4+57n^3+46n^2-8}{3n(10n^4+35n^3+44n^2+16n-6)} \Big/ \left(n+\frac{1}{2n}\right)^{-1} - 1$$
$$= -\frac{16n^6+96n^5+150n^4+39n^3-66n^2+8}{6n^2(10n^4+35n^3+44n^2+16n-6)} < 0,$$

因此

$$f_1(\alpha) \geqslant f_1\left(\frac{n+2}{n+\frac{1}{2n}}\right) = \frac{(2n-1)(32n^5+96n^4+64n^3-41n^2-24n+8)}{(2n^2+1)^2} > 0,$$

$$f_1(\alpha) \geqslant f_1\left(\frac{n+2}{n+\frac{1}{2n}}\right) = \frac{(n+2)(2n-1)(32n^5+16n^4-24n^3-28n^2+15n-2)}{(2n^2+1)^3} > 0,$$

于是 $2\alpha(1-\frac{n\alpha}{n+2}) > \frac{2n-1}{n}\frac{\Delta_3^2}{4d_3^2} \geqslant 0, \forall \alpha \in [\frac{n+2}{n+\frac{1}{2n}}, \frac{n+2}{n})$.

事实上, 除了 λ 项以外, 其余项在 $\alpha \in (1, \frac{n+2}{n})$ 时即为正, 下面证明此事实. 注意到 $d_1 = \frac{n\alpha}{n+2} \cdot \frac{n[3n+6-(n-1)\alpha]}{(2n+1)(n+2)} \geqslant \frac{n\alpha}{n+2} > 0, d_2 = \frac{n\alpha}{n+2} > 0$, 故只需通过计算矩阵 Q_1 的顺序主子式, 验证二次型 $\mathbf{Q}_1$ 的正定性即可:

$$\begin{vmatrix} d_1 & 0 & -\frac{d_4}{2} \\ 0 & d_1 & -\frac{e_4}{2} \\ -\frac{d_4}{2} & -\frac{e_4}{2} & 3 \end{vmatrix} = \frac{n^4\alpha(3-\frac{n-1}{n+2}\alpha)}{2(n+2)(2n+1)^3} f_2(\alpha),$$

其中 $3 - \frac{n-1}{n+2}\alpha > 3 - \frac{n-1}{n+2} \cdot \frac{n+2}{n} = \frac{2n+1}{n} > 0.$

$$\begin{aligned} f_2(\alpha) &= -(37n^2+10n-2)\left(\frac{\alpha}{n+2}\right)^2 + 18(3n+1)\left(\frac{\alpha}{n+2}\right) - 9 \\ &\geqslant \min\left\{f_2(1), f_2(\frac{n+2}{n})\right\} \\ &= \min\left\{\frac{2(4n^2+40n+1)}{(n+2)^2}, \frac{2(2n+1)^2}{n^2}\right\} > 0. \end{aligned}$$

现在, 只需验证 Q_1 的行列式为正即可:

$$\det Q_1 = \frac{n^6\alpha^3(3-\frac{n-1}{n+2}\alpha)^2}{(n+2)^3(2n+1)^4}(1-\frac{n\alpha}{n+2})f_3(\alpha),$$

$$f_3(\alpha) = -2(2n-1)(11n^2+14n+2)\left(\frac{\alpha}{n+2}\right)^2 + (79n^2+58n-2)\left(\frac{\alpha}{n+2}\right) - 27n$$
$$\geqslant \min\left\{f_3(1), f_3(\frac{n+2}{n})\right\} = \min\left\{\frac{2n(4n^2+37n+13)}{(n+2)^2}, \frac{2(n+2)(2n+1)^2}{n^2}\right\} > 0.$$

因此, 当 $\alpha \in (1, \frac{n+2}{n})$ 时, $\mathbf{Q}_1$ 是正定的二次型.

当 $\alpha = \frac{n+2}{n}$ 时, $\Delta_l = \Theta_l = \Xi_l = 0$,

$$d_1 = e_1 = d_2 = e_2 = d_3 = e_3 = 1, \ d_4 = -e_4 = -2, \ \mu = 3, \ \beta = -\frac{2}{n}.$$

微分恒等式变为 Jerison-Lee (4.2) 型恒等式:

$$u^{\frac{2}{n}} \operatorname{Re}\left\{u^{-\frac{2}{n}}\left[\left(\frac{|\nabla_b u|^2}{u} + u^{\frac{n+2}{n}} + \lambda u\right)(D_i + E_i) - n\sqrt{-1}u_0(2D_i - 2E_i + 3G_i)\right]\right\}_{,}^{i}$$

$$= u^{-2}\sum_{i,j,k}|D_{ij}u_{\bar{k}} + E_{\bar{i}k}u_j|^2 + \frac{|\nabla_b u|^2}{u^2}\mathcal{R} + (u^{\frac{2}{n}} + \lambda)\sum_{i,j}(|D_{ij}|^2 + |E_{i\bar{j}}|^2 + \mathcal{R})$$

$$+ \sum_i(|G_i + D_i|^2 + |G_i - E_i|^2 + |G_i|^2).$$

需要以下恒等式帮助:

$$
\begin{aligned}
& u^{-\beta} \operatorname{Re}\Big\{u^{\beta}\Big[\Big(\frac{\alpha}{3}(\frac{1}{2}-\frac{\alpha}{3})\frac{|\nabla_b u|^2}{u^2}-\frac{\alpha}{6}u^{\alpha-1}+(\frac{1}{2}-\frac{\alpha}{3})\lambda\Big)\frac{|\nabla_b u|^2}{u} \\
& +\Big(\frac{1}{2}(\beta+\frac{4}{3}\alpha-1)\frac{|\nabla_b u|^2}{u^2}-u^{\alpha-1}+\lambda-\frac{\sqrt{-1}u_0}{u}\Big)\sqrt{-1}u_0\Big]u_1\Big\}_{,}^{1} \\
= & \operatorname{Re}\Big[\frac{\alpha}{3}(1-\frac{2}{3}\alpha)\frac{|\nabla_b u|^2}{u^2}-\frac{\alpha}{6}u^{\alpha-1}+(\frac{1}{2}-\frac{\alpha}{3})\lambda-\frac{1}{2}(\beta+\frac{4}{3}\alpha-1)\frac{\sqrt{-1}u_0}{u}\Big]D_1 u^1 \\
& +\operatorname{Re}\Big[-\frac{1}{2}(\beta+\frac{4}{3}\alpha-1)\frac{|\nabla_b u|^2}{u^2}+u^{\alpha-1}-\lambda-2\frac{\sqrt{-1}u_0}{u}\Big]G_1 u^1 \\
& -\frac{\alpha}{3}(1-\frac{\alpha}{3})(1-\frac{2}{3}\alpha)\frac{|\nabla_b u|^6}{u^4}.
\end{aligned}
$$

情形 2 的恒等式, $n = 1$:

$$u^{-\beta}\,\mathrm{Re}\left\{u^{\beta}\left[\left(d_1\frac{|\nabla_b u|^2}{u} + d_2 u^{\alpha} + d_3\lambda u\right)D_1 + \sqrt{-1}u_0(d_4 D_1 - \mu G_1)\right]\right\}^{,1}$$

$$= \left(2d_1\frac{|\nabla_b u|^2}{u^2} + d_2 u^{\alpha-1} + d_3\lambda\right)|D_{11}|^2 + \left(d_1\frac{|\nabla_b u|^2}{u^2} + d_2 u^{\alpha-1} + d_3\lambda\right)$$

$$\times\left[2\alpha(1-\frac{\alpha}{3})\frac{|\nabla_b u|^4}{u^2} + \mathcal{R}\right] + \mu|G_1|^2 - d_4\,\mathrm{Re}\,D_1 G^1$$

$$+\,\mathrm{Re}\left[\Delta_1\frac{|\nabla_b u|^2}{u^2} + \Delta_2 u^{\alpha-1} + \Delta_3\lambda + \Delta_4\frac{\sqrt{-1}u_0}{u}\right]D_1 u^1$$

$$+\,\mathrm{Re}\left[\Xi_1\frac{|\nabla_b u|^2}{u^2} + \Xi_2 u^{\alpha-1} + \Xi_3\lambda + \Xi_4\frac{\sqrt{-1}u_0}{u}\right]G_1 u^1.$$

其中的系数如下:

$$\Delta_1 = (\beta + 2\alpha - 2)d_1 + \frac{\alpha}{3}\left(\frac{2}{3}\alpha - 1\right)d_4,\ \Delta_2 = -d_1 + (\beta + 2\alpha - 1)\,d_2 + \frac{\alpha}{3}d_4,$$

$$\Delta_3 = d_1 + (\beta + \alpha)\,d_3 + \left(\frac{2}{3}\alpha - 1\right)d_4,\ \Delta_4 = d_1 + \left(\beta + \frac{5}{3}\alpha - 1\right)d_4 + \frac{\alpha}{3}\left(\frac{2}{3}\alpha - 1\right)\mu,$$

$$\Xi_1 = 3d_1 - \frac{\alpha}{3}\left(\frac{2}{3}\alpha - 1\right)\mu,\ \Xi_2 = 3d_2 - \frac{\alpha}{3}\mu,\ \Xi_3 = 3d_3 - \left(\frac{2}{3}\alpha - 1\right)\mu,\ \Xi_4 = 3d_4 - \beta\mu.$$

取 $d_1 = \frac{\alpha}{36}(5\alpha - 3), d_2 = d_3 = \frac{\alpha - 1}{2}, d_4 = 2 - \frac{4}{3}\alpha, \mu = 3, \beta = 1 - \alpha$, 并将两个恒等式取合适的线性组合:

$$
\begin{aligned}
& u^{\alpha-1}\,\mathrm{Re}\left\{u^{1-\alpha}\left[\left(\frac{\alpha}{36}(5\alpha-3)\frac{|\nabla_b u|^2}{u} + \frac{\alpha-1}{2}(u^\alpha + \lambda u)\right)D_1 + \sqrt{-1}u_0\left((2-\frac{4}{3}\alpha)D_1\right.\right.\right. \\
& \left. - 3G_1\right) + \frac{3-\alpha}{2}\left(\frac{\alpha}{3}(\frac{1}{2} - \frac{\alpha}{3})\frac{|\nabla_b u|^2}{u^2} - \frac{\alpha}{6}u^{\alpha-1} + (\frac{1}{2} - \frac{\alpha}{3})\lambda\right)\frac{|\nabla_b u|^2}{u}u_1 \\
& \left.\left. + \frac{3-\alpha}{2}\left(\frac{\alpha}{6}\frac{|\nabla_b u|^2}{u^2} - u^{\alpha-1} + \lambda - \frac{\sqrt{-1}u_0}{u}\right)\sqrt{-1}u_0 u_1\right]\right\}_{,1} \\
= & \frac{\alpha-1}{2}\lambda\left[\left|D_{11} + \frac{7}{12}(3-\alpha)\frac{u_1 u_1}{u}\right|^2 + \frac{1}{144}(3-\alpha)(145\alpha - 147)\frac{|\nabla_b u|^4}{u^2} + \mathcal{R}\right] \\
& + \frac{\alpha-1}{2}u^{\alpha-1}\left[|D_{11}|^2 + 2\alpha(1-\frac{\alpha}{3})\frac{|\nabla_b u|^4}{u^2} + \mathcal{R}\right] + \frac{1}{12}(\alpha-1)(\alpha+3)\frac{|\nabla_b u|^2}{u^2}\mathcal{R} + \mathbf{Q}_2,
\end{aligned}
$$

$$\mathbf{Q}_2 = \frac{\alpha}{18}(5\alpha - 3)|D_1|^2 + 3|G_1|^2 + (\frac{4}{3}\alpha - 2)\operatorname{Re} D_1 G^1 + (3-\alpha)\frac{|\nabla_b u|^2}{u^2} \times \left[\frac{\alpha}{108}(5\alpha - 3)\operatorname{Re} D_1 u^1 + \frac{\alpha}{6}\operatorname{Re} G_1 u^1 + \frac{\alpha}{18}(\alpha - 1)(\alpha + 3)\frac{|\nabla_b u|^4}{u^2}\right],$$

$$Q_2 = \begin{pmatrix} \frac{\alpha}{18}(5\alpha - 3) & \frac{2}{3}\alpha - 1 & \frac{\alpha}{216}(3-\alpha)(5\alpha - 3) \\ \frac{2}{3}\alpha - 1 & 3 & \frac{\alpha}{12}(3-\alpha) \\ \frac{\alpha}{216}(3-\alpha)(5\alpha-3) & \frac{\alpha}{12}(3-\alpha) & \frac{\alpha}{18}(3-\alpha)(\alpha-1)(\alpha+3) \end{pmatrix}.$$

- 注: 当 $\alpha = 3$, 依旧会回到 Jerison-Lee (4.2) 型恒等式在 $n = 1$ 的情况.

情形 3: $1.06 \leqslant \alpha < 3, n = 1$

先检查 λ 项的正性: 当 $\alpha \in [1.06, 3)$ 时, $\alpha - 1 > 0$, $(3-\alpha)(145\alpha - 147) > 0$.

验证矩阵 Q_2 在 $\alpha \in [1.06, 3)$ 的正定性: $\frac{\alpha}{18}(5\alpha - 3) > \frac{\alpha}{9} > 0$,

$$\begin{vmatrix} \frac{\alpha}{18}(5\alpha-3) & \frac{2}{3}\alpha - 1 \\ \frac{2}{3}\alpha - 1 & 3 \end{vmatrix} = \frac{1}{18}(\alpha+3)(7\alpha-6) > 0,$$

$$\det Q_2 = \frac{\alpha}{5184}(3-\alpha)(3+\alpha)f_4(\alpha),$$

其中 $f_4(\alpha) = 117\alpha^3 + 110\alpha^2 - 519\alpha + 288$, 研究单调性:

$$f_4'(\alpha) = 351\alpha^2 + 220\alpha - 519 > 351 + 220 - 519 = 52 > 0,$$

那么 $f_4(\alpha) \geqslant f_4(1.06) = 0.804872 > 0$. 于是当 $\alpha \in [1.06, 3)$ 时, 二次型 $\mathbf{Q}_2$ 正定.

$$
\begin{aligned}
& u^{-\frac{1}{2}} \operatorname{Re} \Bigg\{ u^{\frac{1}{2}} \Bigg[\left(\frac{1}{18} \frac{|\nabla_b u|^2}{u} + \frac{\alpha-1}{2} (u^\alpha + \lambda u) \right) D_1 + \sqrt{-1} u_0 \left(\frac{2}{3} D_1 - 3 G_1 \right) \\
& + \frac{3-\alpha}{2} \left[\left(-\frac{\alpha}{6} u^{\alpha-1} + (\frac{1}{2} - \frac{\alpha}{3}) \lambda \right) \frac{|\nabla_b u|^2}{u} + \left(-u^{\alpha-1} + \lambda - \frac{\sqrt{-1} u_0}{u} \right) \sqrt{-1} u_0 \right] u_1 \\
& + \left((2\alpha - 3)(\frac{1}{36}\alpha^2 - \frac{1}{12}\alpha + \frac{1}{40}) \frac{|\nabla_b u|^2}{u} - (\frac{1}{3}\alpha^2 - \frac{9}{8}\alpha + \frac{3}{5}) \sqrt{-1} u_0 \right) \frac{|\nabla_b u|^2}{u^2} u_1 \Bigg] \Bigg\}^1_, \\
= & \frac{\alpha-1}{2} u^{\alpha-1} \left[\left| D_{11} + \frac{1}{36} (39 - 7\alpha) \frac{u_1 u_1}{u} \right|^2 - \frac{11925\alpha^2 - 15690\alpha + 3161}{6480} \frac{|\nabla_b u|^4}{u^2} \right] \\
& + \frac{\alpha-1}{2} \lambda \left[\left| D_{11} + \frac{1}{9} (6\alpha + 1) \frac{u_1 u_1}{u} \right|^2 - \frac{90\alpha^2 - 150\alpha + 1}{81} \frac{|\nabla_b u|^4}{u^2} \right] \\
& + \left[\frac{1}{18} \frac{|\nabla_b u|^2}{u^2} + \frac{\alpha-1}{2} (u^{\alpha-1} + \lambda) \right] \mathcal{R} + \mathbf{Q}_3,
\end{aligned}
$$

情形 4: $1 < \alpha \leqslant 1.06, n = 1$

$$\mathbf{Q}_3 = \frac{1}{9}|D_1|^2 + 3|G_1|^2 - \frac{2}{3}\operatorname{Re} D_1 G^1 + \frac{9}{20}\frac{|\nabla_b u|^2}{u^2}u_0^2 + \Delta_4' \operatorname{Re}\frac{\sqrt{-1}u_0}{u}D_1 u^1$$
$$+ \Xi_4' \operatorname{Re}\frac{\sqrt{-1}u_0}{u}G_1 u^1 + \Delta_1'\frac{|\nabla_b u|^2}{u^2}\operatorname{Re} D_1 u^1 + \Xi_1'\frac{|\nabla_b u|^2}{u^2}\operatorname{Re} G_1 u^1 + A\frac{|\nabla_b u|^6}{u^4},$$

$$Q_3 = \begin{pmatrix} \frac{1}{9} & -\frac{1}{3} & \frac{\Delta_4'}{2} & \frac{\Delta_1'}{2} \\ -\frac{1}{3} & 3 & \frac{\Xi_4'}{2} & \frac{\Xi_1'}{2} \\ \frac{\Delta_4'}{2} & \frac{\Xi_4'}{2} & \frac{9}{20} & 0 \\ \frac{\Delta_1'}{2} & \frac{\Xi_1'}{2} & 0 & A \end{pmatrix},$$

$$\Delta_1' = \frac{1}{270}(30\alpha^3 - 95\alpha^2 + 132\alpha - 63),\ \Delta_4' = \frac{1}{360}(360\alpha^2 - 365\alpha + 116),$$

$$\Xi_1' = -\frac{1}{120}(40\alpha^2 + 15\alpha - 92),\ \Xi_4' = \alpha - \frac{5}{2},$$

$$A = \frac{1}{2160}(80\alpha^4 - 600\alpha^3 + 1468\alpha^2 - 1272\alpha + 405).$$

当 $\alpha \in (1, 1.06]$ 时, 分别检查 $u^{\alpha-1}\frac{|\nabla_b u|^4}{u^2}$ 项和 $\lambda\frac{|\nabla_b u|^4}{u^2}$ 项的正性:

$$-(4565\alpha^2 - 15690\alpha + 10521) \geqslant -(4565 \times 1 - 15690 \times 1 + 10521) = 604 > 0,$$

$$-(90\alpha^2 - 150\alpha + 1) \geqslant -(90 \times 1.06^2 - 150 + 1) = 47.876 > 0.$$

计算矩阵 Q_3 的顺序主子式:

$$\begin{vmatrix} \frac{1}{9} & -\frac{1}{3} \\ -\frac{1}{3} & 3 \end{vmatrix} = \frac{2}{9} > 0, \quad \begin{vmatrix} \frac{1}{9} & -\frac{1}{3} & \frac{\Delta_4'}{2} \\ -\frac{1}{3} & 3 & \frac{\Xi_4'}{2} \\ \frac{\Delta_4'}{2} & \frac{\Xi_4'}{2} & \frac{9}{20} \end{vmatrix} = \frac{f_5(\alpha)}{57600}, \quad \det Q_3 = \frac{f_6(\alpha)}{29859840000},$$

其中 $f_5(\alpha) = -43200\alpha^4 + 78000\alpha^3 - 40115\alpha^2 + 8800\alpha - 992$,

$$\begin{aligned} f_6(\alpha) = &- 460800000\alpha^8 + 6320640000\alpha^7 - 25055552000\alpha^6 \\ &+ 44595172000\alpha^5 - 42848423575\alpha^4 + 24660626800\alpha^3 \\ &- 8756098960\alpha^2 + 1823449600\alpha - 252801536. \end{aligned}$$

$$
\begin{aligned}
f_5''(\alpha) =&10(-51840\alpha^2 + 46800\alpha - 8023)\\
&<10(-51840 + 46800 \times 1.06 - 8023) = -102550 < 0,
\end{aligned}
$$

于是 f_5 是凹函数, 进而 $f_5(\alpha) \geqslant \min\{f_5(1), f_5(1.06)\} = \min\{2493, 1623.03\} > 0.$

仿照上述过程, 检查 f_6 的凹性:

$$
\begin{aligned}
f_6''(\alpha) = &-20(1290240000\alpha^6 - 13273344000\alpha^5 + 37583328000\alpha^4\\
&- 44595172000\alpha^3 + 25709054145\alpha^2 - 7398188040\alpha + 875609896),
\end{aligned}
$$

$$
\begin{aligned}
f_6^{(3)}(\alpha) = &-600(258048000\alpha^5 - 2212224000\alpha^4 + 5011110400\alpha^3\\
&- 4459517200\alpha^2 + 1713936943\alpha - 246606268),
\end{aligned}
$$

$$
\begin{aligned}
f_6^{(4)}(\alpha) = &-600(1290240000\alpha^4 - 8848896000\alpha^3\\
&+ 15033331200\alpha^2 - 8919034400\alpha + 1713936943),
\end{aligned}
$$

$$f_6^{(5)}(\alpha) = -480000(6451200\alpha^3 - 33183360\alpha^2 + 37583328\alpha - 11148793),$$

$$\begin{aligned} f_6^{(6)}(\alpha) =& 46080000(-201600\alpha^2 + 691320\alpha - 391493) \\ \geqslant& f_6^{(6)}(1) = 46080000 \times 98227 > 0, \end{aligned}$$

$$f_6^{(5)}(\alpha) \geqslant f_6^{(5)}(1) = 480000 \times 297625 > 0,$$

$$f_6^{(4)}(\alpha) \leqslant f_6^{(4)}(1.06) = -600 \times 240932969.8544 < 0,$$

$$f_6^{(3)}(\alpha) \leqslant f_6^{(3)}(1) = -600 \times 64747875 < 0,$$

$$f_6''(\alpha) \leqslant f_6''(1) = -20 \times 191528001 < 0,$$

于是 f_6 是凹函数, 进而

$$f_6(\alpha) \geqslant \min\{f_6(1), f_6(1.06)\} = \min\{26212329, 2.38 \times 10^7\} > 0.$$

于是当 $\alpha \in (1, 1.06]$ 时, 二次型 $\mathbf{Q}_3$ 正定.

1. 闭流形上的半线性方程与不变张量技术
2. 闭 CR 流形上的（次）临界指标方程
3. Heisenberg 群上的（次）临界指标方程

$$\Delta u + u^{\alpha} = 0,\ \ u > 0,\ \ \mathbb{H}^n.$$

若 $1 < \alpha < \dfrac{n+2}{n}$, 无解; 若 $\alpha = \dfrac{n+2}{n}$ 且 $u \in L^{2+\frac{2}{n}}(\mathbb{H}^n)$, 则存在 $\lambda \in \mathbb{C}, \mu \in \mathbb{C}^n$, $\operatorname{Im}\lambda > \dfrac{|\mu|^2}{4}$, 使得

$$u(z,t) = c_{n,\lambda,\mu}\left|t + \sqrt{-1}|z|^2 + z\cdot\mu + \lambda\right|^{-n},$$

方程的由来: CR-Yamabe 问题, Folland-Stein 不等式的最佳常数.

- D. Jerison, John. M. Lee, 1988, J. Amer. Math. Soc.: $\alpha = \frac{n+2}{n}$ 且 $u \in L^{2+\frac{2}{n}}(\mathbb{H}^n)$. 在这篇论文中, 他们使用计算机得到了三个微分恒等式, 并提出了如下问题:
  1. 为何会存在三个线性无关的微分恒等式?
  2. 是否存在一套理论框架或系统性的算法, 能够找出有用的微分恒等式?
- 麻希南, 欧乾忠, 2023, Adv. Math.: $1 < \alpha < \frac{n+2}{n}$.
- 注: D. Jerison, John. M. Lee 也在 1988 年解决了 CR 球面 Yamabe 方程的解的分类, 这个过程与 Heisenberg 群的讨论过程相差一个 Cayley 变换.

以下内容摘自 David Jerison and John M. Lee. Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. J. Amer. Math. Soc., 1(1):1 13, 1988.

- Our approach was to write the most general such formula with undetermined coefficients, in which the tensors on the right-hand side are formed from combinations of B, A, and div A. Equating like terms leads to a system of linear equations for the coefficients. One seeks a solution for which the right-hand side vanishes identically. This approach led us to a 25×25 variable-coefficient system which we solved using the computer algebra program MACSYMA. Surprisingly, we then found a three-dimensional family of solutions with positive right-hand side.

- An interesting (but vaguely defined) problem raised by this work is to find an ”explanation” for the existence of divergence formulas. Is there a theorectical framework that would predict the existence and the structure of such formulas, so that they could be discovered more systematically?
- As we mentioned in the introduction, the reason for the existence of these formulas is a mystery.

针对 Jerison-Lee 之问, 我们给出了肯定的回答: 所有合理的六阶导数恒等式, 只能是 Jerison-Lee 提出的恒等式的线性组合, 而 Jerison-Lee 寻求的理论框架就是不变张量技术!

$$
\begin{aligned}
&u^{\frac{2}{n}} \operatorname{Re}\left\{u^{-\frac{2}{n}}\left[\left(\frac{|\nabla_b u|^2}{u}+u^{\frac{n+2}{n}}\right)(D_i+E_i)-n\sqrt{-1}u_0(2D_i-2E_i+3G_i)\right]\right\}_{,\bar{i}} \\
=&u^{\frac{2}{n}}\sum_{i,j}(|D_{ij}|^2+|E_{i\bar{j}}|^2)+\sum_i(|G_i|^2+|G_i+D_i|^2+|G_i-E_i|^2)+u^{-2}\sum_{i,j,k}|D_{ij}u_{\bar{k}}+E_{i\bar{k}}u_j|^2.
\end{aligned}
$$

这是用来做正解分类定理的恒等式.

- 注: 原论文的计算做了变换 $u=e^{nf}$, 这里用没做变换的语言写出.

$$
\begin{aligned}
& u^{\frac{2}{n}} \operatorname{Re}\Bigg\{u^{-\frac{2}{n}}\Bigg\{\Big(nu^{\frac{n+2}{n}}-2n^2\sqrt{-1}u_0\Big)D_i+\Big((n+2)\frac{|\nabla_b u|^2}{u}+u^{\frac{n+2}{n}}+2n\sqrt{-1}u_0\Big)E_i \\
& -(n+2)n\sqrt{-1}u_0G_i+n\Big[D_ju_{\bar{j}}-E_ju_{\bar{j}}+\frac{n-1}{n^2}\Big(\frac{|\nabla_b u|^4}{u^2}+u^{\frac{2}{n}}|\nabla_b u|^2 \\
& \quad -n\frac{|\nabla_b u|^2\cdot n\sqrt{-1}u_0}{u}+(n+1)u^{\frac{n+2}{n}}\cdot n\sqrt{-1}u_0-(n+1)n^2u_0^2\Big)\Big]\frac{u_i}{u}\Bigg\}\Bigg\}_{,\bar{i}} \\
= & (n+2)\frac{|\nabla_b u|^2}{u^2}\sum_{i,j}|E_{i\bar{j}}|^2+\sum_i|E_i|^2+(n-2)\sum_i|D_i|^2+(n+1)\sum_i|G_i+D_i|^2 \\
& +\sum_i|G_i-D_i-E_i|^2+u^{\frac{2}{n}}\sum_{i,j}(|E_{i\bar{j}}|^2+n|D_{ij}|^2).
\end{aligned}
$$

这个恒等式也是正定的.

$$
\begin{aligned}
&u^{\frac{2}{n}} \operatorname{Re}\Big\{u^{-\frac{2}{n}}\Big\{\Big(\frac{|\nabla_b u|^2}{u}+u^{\frac{n+2}{n}}\Big)(D_i-2E_i)\\
&\qquad -n\sqrt{-1}u_0[(3n-1)D_i-(3n+2)E_i+3nG_i]\Big\}\Big\}_{,\bar{i}}\\
=&\Big[\frac{|\nabla_b u|^2}{u^2}+u^{\frac{2}{n}}\Big]\sum_{i,j}(|D_{ij}|^2-2|E_{i\bar{j}}|^2)+\sum_i(|D_i|^2-2|E_i|^2+3n|G_i|^2)\\
&-\operatorname{Re}D_iE_{\bar{i}}+(3n-1)\operatorname{Re}D_iG_{\bar{i}}-(3n+2)\operatorname{Re}E_iG_{\bar{i}}.
\end{aligned}
$$

这个恒等式虽然不是正定的,但是可以用它对前两个恒等式做小扰动,在不破坏正定性的情况下得到新的恒等式.

Proposition (麻希南-欧乾忠-吴天, 2023, arXiv: 2311.16428v2)

当 $n \geqslant 2$ 时, 所有正定的 $\{(0,0),2,6,+\}$ 型恒等式为

$$u^{\frac{2}{n}}\,\mathrm{Re}\left\{u^{-\frac{2}{n}}\left\{\left(\left(d_1\frac{|\nabla_b u|^2}{u}+(d_1+a)u^{\frac{n+2}{n}}+(d_1-\frac{n-2}{n}a-\mu)n\sqrt{-1}u_0\right)D_i\right.\right.\right.$$

$$+\left(\frac{(n+2)(d_1+a)-\mu}{n-1}\frac{|\nabla_b u|^2}{u}+\frac{(n+2)d_1+(2+\frac{1}{n})a-\mu}{n-1}u^{\frac{n+2}{n}}\right.$$

$$\left.+\frac{-(n+2)d_1-(n+\frac{2}{n})a+n\mu}{n-1}\cdot n\sqrt{-1}u_0\right)E_i-\mu n\sqrt{-1}u_0G_i$$

$$+a\left[D_ju_{\bar{j}}-E_ju_{\bar{j}}+\frac{n-1}{n^2}\left(\frac{|\nabla_b u|^4}{u^2}+u^{\frac{2}{n}}|\nabla_b u|^2-n\frac{|\nabla_b u|^2\cdot n\sqrt{-1}u_0}{u}\right.\right.$$

Proposition (麻希南-欧乾忠-吴天, 2023, arXiv: 2311.16428v2)

$$+ (n+1)u^{\frac{n+2}{n}} \cdot n\sqrt{-1}u_0 - (n+1)n^2u_0^2\Big)\Big]\frac{u_i}{u}\Big\}\Big\}_{,\bar{i}}$$

$$= \left[d_1\frac{|\nabla_b u|^2}{u^2} + (d_1+a)u^{\frac{2}{n}}\right]\sum_{i,j=1}^{n}|D_{ij}|^2 + (d_1+2a)\sum_{i=1}^{n}|D_i|^2$$

$$+ \left[\frac{(n+2)(d_1+a)-\mu}{n-1}\frac{|\nabla_b u|^2}{u^2} + \frac{(n+2)d_1+(2+\frac{1}{n})a-\mu}{n-1}u^{\frac{2}{n}}\right]\sum_{i,j=1}^{n}|E_{i\bar{j}}|^2$$

$$+ \frac{(n+2)d_1+3a-\mu}{n-1}\sum_{i=1}^{n}|E_i|^2 + \mu\sum_{i=1}^{n}|G_i|^2 + \frac{(2n+1)d_1+3a-\mu}{n-1}\operatorname{Re} D_iE_{\bar{i}}$$

Proposition (麻希南-欧乾忠-吴天, 2023, arXiv: 2311.16428v2)

$$+\left(-d_1+\frac{n-2}{n}a+\mu\right)\operatorname{Re}D_iG_{\bar{i}}+\frac{(n+2)d_1+(n+\frac{2}{n})a-n\mu}{n-1}\operatorname{Re}E_iG_{\bar{i}}.$$

其中参数 d_1, a, μ 满足

$$d_1 \geqslant \max\{0,-a\},\ (n+2)d_1-\mu \geqslant \max\left\{-(n+2)a,-(2+\frac{1}{n})a\right\}, \tag{1}$$

Proposition (麻希南-欧乾忠-吴天, 2023, arXiv: 2311.16428v2)

以及三阶对称矩阵 B 半正定, 其中矩阵 B 满足

$$B_{11}=\mu,\ B_{12}=\frac{1}{2}(-d_1+\frac{n-2}{n}a+\mu),$$

$$B_{13}=\frac{(n+2)d_1+(n+\frac{2}{n})a-n\mu}{2(n-1)},\ B_{22}=2(d_1+a),$$

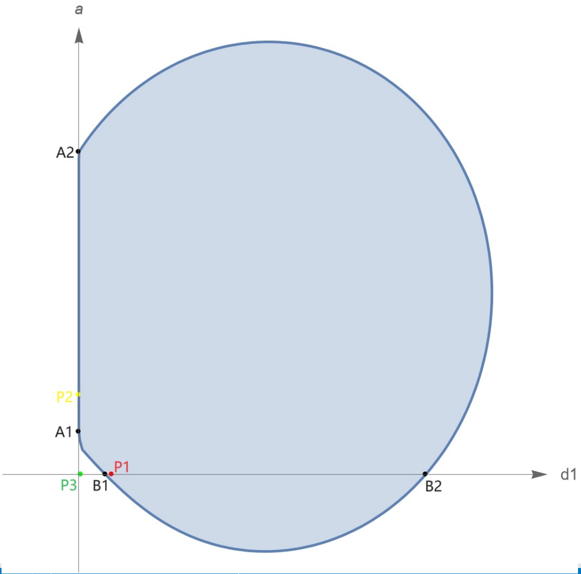
$$B_{23}=\frac{(2n+1)d_1+3a-\mu}{2(n-1)},\ B_{33}=\frac{2n-1}{(n-1)^2}\left[(n+2)d_1+\frac{n^2+5n-3}{2n-1}a-\mu\right].$$

常数 d_1, a, μ 在线性相关意义下, 决定了如 Jerison-Lee 所说的三维恒等式族.

- $d_1 = 1, a = 0, \mu = 3$, 对应于 Jerison-Lee 恒等式 (4.2)
- $d_1 = 0, a = \frac{3n}{n+2}, \mu = 3$, 对应于 Jerison-Lee 恒等式 (4.3)
- $d_1 = \frac{1}{n}, a = 0, \mu = 3$ 对应于 Jerison-Lee 恒等式 (4.4)
- d_1, a, μ 成比例的情况下是同一个恒等式

当 $n = 1$ 时, 命题中的恒等式在线性相关意义下只有一个自由维度, 而三个 Jerison-Lee 恒等式也退化为同一个.

取定 $\mu = 3$ 的正定性范围



取定 $\mu=3$ 的正定性范围

$$\mathrm{P1}=(1,0),\quad \mathrm{P2}=(0,\frac{3n}{n+2}),\quad \mathrm{P3}=(\frac{1}{n},0),\quad \mathrm{A1}=(0,\frac{3n}{2n+1}),$$

$$\begin{aligned}\mathrm{A2}=\Bigg(0,\ &\frac{2\sqrt{n(73n^7+538n^6+1435n^5+134n^4-1439n^3-120n^2+292n+48)}}{3n^4-2n^3-5n^2+26n+8}\\ &\times\cos\Bigg\{\frac{1}{3}\arccos\Big[\sqrt{n}(595n^{10}+7017n^9+30666n^8+55019n^7-7692n^6-82095n^5\\ &\qquad-12345n^4+38598n^3+2556n^2-6920n-1440)\\ &\qquad\times(73n^7+538n^6+1435n^5+134n^4-1439n^3-120n^2+292n+48)^{-3/2}\Big]\Bigg\}\\ &+\frac{n(10n^3+35n^2+4)}{(n+2)(3n^3-8n^2+11n+4)}\Bigg),\end{aligned}$$

取定 $\mu = 3$ 的正定性范围

$$\mathrm{B1} = \left(\frac{\sqrt{468n^4 + 1380n^3 + n^2 - 1500n + 612}}{3n^2 + 8n + 4} \cos\frac{\theta - 2\pi}{3} + \frac{24n^2 + 43n - 18}{2(n+2)(3n+2)}, 0 \right),$$

$$\mathrm{B2} = \left(\frac{\sqrt{468n^4 + 1380n^3 + n^2 - 1500n + 612}}{3n^2 + 8n + 4} \cos\frac{\theta}{3} + \frac{24n^2 + 43n - 18}{2(n+2)(3n+2)}, 0 \right),$$

$$\theta = \arccos \frac{9936n^6 + 44172n^5 + 32202n^4 - 66149n^3 - 35622n^2 + 54756n - 15336}{(468n^4 + 1380n^3 + n^2 - 1500n + 612)^{\frac{3}{2}}},$$

回忆情形 2 的恒等式:

$$
u^{-\beta}\operatorname{Re}\left\{u^{\beta}\left[\left(d_1\frac{|\nabla_b u|^2}{u}+d_2u^{\alpha}+d_3\lambda u\right)(D_i+E_i)+n\sqrt{-1}u_0(d_4D_i+e_4E_i-3G_i)\right]\right\}_{,}^{\ \ i}
$$

$$
\begin{aligned}
=&d_1u^{-2}\sum_{i,j,k}|D_{ij}u_{\bar k}+E_{i\bar k}u_j|^2+d_2u^{\alpha-1}\sum_{i,j}\left[|D_{ij}|^2+|E_{i\bar j}|^2+2\alpha(1-\frac{n\alpha}{n+2})\frac{|\nabla_b u|^4}{u^2}\right]\\
&+d_3\lambda\sum_{i,j}\left[\left|D_{ij}+\frac{\Delta_3}{2d_3}\frac{u_iu_j}{u}\right|^2+\left|E_{i\bar j}+\frac{\Delta_3}{2d_3}L_{i\bar j}\right|^2+\left(2\alpha(1-\frac{n\alpha}{n+2})-\frac{2n-1}{n}\frac{\Delta_3^2}{4d_3^2}\right)\frac{|\nabla_b u|^4}{u^2}\right]\\
&+\left[d_1\frac{|\nabla_b u|^2}{u^2}+d_2u^{\alpha-1}+d_3\lambda\right]\mathcal{R}+\mathbf{Q}_1,
\end{aligned}
$$

$\Delta_b u + u^\alpha = 0, \mathbb{H}^n$, 若 $1 < \alpha < \dfrac{n+2}{n}$, 方程无正解.

将 CR 几何情形 2 的恒等式限制在 $\mathbb{H}^n$ 上: 存在 $\delta > 0$,

$$u^{-\beta}\,\mathrm{Re}\left\{u^\beta\left[\left(d_1\frac{|\nabla_b u|^2}{u}+d_2u^\alpha\right)(D_i+E_i)+n\sqrt{-1}u_0(d_4D_i+e_4E_i-3G_i)\right]\right\}_{,\bar{i}}$$

$$=d_1u^{-2}\sum_{i,j,k=1}^n|D_{ij}u_{\bar{k}}+E_{\bar{i}k}u_j|^2+d_2u^{\alpha-1}\sum_{i,j=1}^n\left[|D_{ij}|^2+|E_{i\bar{j}}|^2+2\alpha\left(1-\frac{n\alpha}{n+2}\right)\frac{|\nabla_b u|^4}{u^2}\right]+\mathbf{Q}_1$$

$$\geqslant\delta u^{\alpha-1}\left[\sum_{i,j=1}^n(|D_{ij}|^2+|E_{i\bar{j}}|^2)+\frac{|\nabla_b u|^4}{u^2}\right]+\delta\left[\sum_{i=1}^n(|D_i|^2+|E_i|^2+|G_i|^2)+\frac{|\nabla_b u|^6}{u^4}\right].$$

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谢谢大家！

吴天

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